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I. SOLUTION OF MAXWELL'S EQUATIONS FOR SYMMETRIC SYSTEMS

Solving Maxwell's differential equations corresponds to solving a system of coupled differential equations to determine two unknown vector functions of four variables. This is no easy task in the general case. Fortunately, as with most problems in physics, the existence of *symmetries* in the system of interest, in the geometry and distribution of charge and current in space, simplifies the solution.

A. Symmetries, coordinate systems, and differential equations

The first thing that you do when faced with a problem in electromagnetism is to look carefully at the system and ask yourself: *does this system have some kind of symmetry?* If the system remains exactly the same under some transformation, such as a translation in space or time, a rotation about a given axis or point, etc, then the electric and magnetic fields should also remain the same under this transformation: they must reflect the symmetry of the system that they describe. You can be assured that this property of the fields, dictated by the physics, is also reflected by the mathematical solution of Maxwell's equations. This solution for appropriate boundary conditions is *unique*. Therefore, if you guess one solution using your physical intuition and the symmetry properties, then you know that this is the *one and only* solution. In the case of symmetry, you can make an assumption about the $\mathbf{r}$–dependence of the unknown electric and magnetic fields, which reflects the fact that the fields remain the same under the symmetry transformation, and solve the equations+boundary conditions to verify that your assumption is correct.

In symmetric systems, it is advantageous to solve Maxwell's equations using the coordinate system related to the particular symmetry. For example, in the case of spherical symmetry around point O, the fields only depend on the distance r from O, not on any angles. It is thus advantageous to work with spherical coordinates, $\mathbf{r} = (r, \theta, \phi)$, with origin at O. Due to spherical symmetry, the dependence of the unknown field functions on three variables x, y, z reduces to a dependence on *one* variable $r = \sqrt{x^2 + y^2 + z^2}$. You can safely assume the functional form

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(r, t) \ , \ \mathbf{B}(x, y, z, t) = \mathbf{B}(r, t),$$

analyze the fields in the spherical coordinate system, $\mathbf{E}(r) = (E_r(r), E_\theta(r), E_\phi(r))$ and $\mathbf{B}(r) = (B_r(r), B_\theta(r), B_\phi(r))$, substitute this ansatz in Maxwell's differential equations, and use the mathematical relations in the cover of Griffiths to express the derivatives ∇ in spherical coordinates. Noting that

$$\frac{\partial \mathbf{E}}{\partial \phi} = \frac{\partial \mathbf{E}}{\partial \theta} = \frac{\partial \mathbf{B}}{\partial \phi} = \frac{\partial \mathbf{B}}{\partial \theta}\mathbf{B} = 0,$$

we thus obtain that

$$\nabla \cdot \mathbf{E}(r) = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 E_r(r)] + \frac{1}{r}\cot\theta E_\theta(r)$$

and

$$\nabla \times \mathbf{E}(r) = \frac{\cot\theta}{r}E_\phi(r)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}[rE_\phi(r)]\hat{\theta} + \frac{1}{r}\frac{\partial}{\partial r}[rE_\theta(r)]\hat{\phi}$$

and similar for the magnetic field. Similarly, in the case of cylindrical symmetry around a line, usually taken as the axis z, the fields only depend on the distance r perpendicular to this line. It is then advantageous to express Maxwell's equations in the cylindrical coordinate system and calculate the derivatives ∇ after noting that

$$\frac{\partial \mathbf{E}}{\partial \phi} = \frac{\partial \mathbf{E}}{\partial z} = \frac{\partial \mathbf{B}}{\partial \phi} = \frac{\partial \mathbf{B}}{\partial z} = 0.$$

We thus obtain from the expressions of the vector derivatives in the cylndrical coordinate system found in the cover of Griffith's book that

$$\nabla \cdot \mathbf{E}(r) = \frac{1}{r}\frac{\partial}{\partial r}[rE_r(r)]$$

and

$$\nabla \times \mathbf{E}(r) = -\frac{\partial E_z(r)}{\partial r}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial r}[rE_\phi(r)]\hat{z}$$

and similar for the magnetic field. The above expressions reduce Maxwell's equations from partial to ordinary differential equations. Note that some systems may remain invariant only under rotation around the axis z but not when translated along the z–axis. In this case, we can make the assumption that

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(r, z) \ , \ \mathbf{B}(\mathbf{r}) = \mathbf{B}(r, z)$$

and obtain partial differential equations with two variables insetad of three. In the case of a system that remains the same under any translational within the x–y plane, the fields only depend on z:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(z) \ , \ \mathbf{B}(\mathbf{r}) = \mathbf{B}(z).$$

If furthermore the system is invariant under reflection about a symmetry x–y plane, then the fields also reflect this:

$$E_z(z) = -E_z(-z) \ , \ B_z(z) = -B_z(-z).$$

An important simplification arises if the geometry and charge and current distributions do not change with time. This does not mean that charge does not move, it means that the resulting current remains the same at all times. In this case,

$$\frac{\partial}{\partial t}\mathbf{E} = \frac{\partial}{\partial t}\mathbf{B} = 0.$$

The electric and magnetic fields *decouple* in Maxwell's equations, which can be solved independently of each other: solve two equations to obtain the electric field,

$$\nabla \times \mathbf{E} = 0 \ , \ \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

then solve two equations

$$\nabla \cdot \mathbf{B} = 0 \ , \ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

to get the magnetic field. Note that, in the absence of any charge distributions (e.g. neutral conductors), the assumption **E**=0 satisfies the differential equations and the boundary conditions and is therefore the unique solution. Stationary currents do not create electric fields, only magnetic fields. This is not the case however for time–dependent currents, where the electric and magnetic fields couple in Maxwell's equations. Similarly, in the absence of any currents, **B**=0 satisfies the differential equations and boundary conditions and is therefore the unique solution. Time–independent charge distributions do not produce magnetic fields, unlike for time–dependent distributions.

B. Boundary Conditions

Maxwell's differential equations do not have a unique solution unless we also consider appropriate boundary conditions. The latter correspond to making sure that the behavior of the solution at infinity and special points such as r=0 is physical. For example, at r=0 we make sure that the fields do not diverge unless there is a point or line charge or line current there, in which case the solutions of Maxwell's equations for $r \rightarrow 0$ should recover the known fields of these charges or currents. Furthermore, when we view the system from infinity, it often looks like a point or a surface since the viewing distance is much larger than the characteristic dimensions. In this limit, the solution of Maxwell's equations should again recover the result expected for zero characteristic dimensions of the charge and current distributions.

In addition, when we have charge or current restricted within a surface, we approximate the rapid variation of the fields inside the small surface depth, which we consider as approximately zero, by a discontinuity in the fields. These discontinuities are derived from Maxwell's equations in Griffiths Vol 1, pages 115-117 (electric field) and pages 305-307 (magnetic field). In particular, when a surface with surface charge σ at a given point separates two regions of space, which we call 1 and 2, then the electric field components parallel to the surface are continuous while the electric field components perpendicular to the surface are discontinuous across the surface. The opposite holds for the magnetic field components. To describe the discontinuity, which results if we approximate the volume charge or current density by a surface density and neglect the surface depth, we define the unit vector $\mathbf{n}$ perpendicular to the surface at the given point and pointing from region 1 towards region 2. We then obtain for every point on the surface the following relations between the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ right below and right above the surface, in regions 1 and 2 respectively:

$$\mathbf{E}_2 \cdot \hat{\mathbf{n}} - \mathbf{E}_1 \cdot \hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0} \tag{1}$$

and

$$\mathbf{E}_{2\|} = \mathbf{E}_{1\|} \tag{2}$$

where $\mathbf{E}_{\|}$ denote the projection of the electric field on the surface at the given point. More compactly,

$$\mathbf{E}_2 - \mathbf{E}_1 = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \tag{3}$$

Similarly, Maxwell's equations give for the magnetic field components right below and right above the surface

$$\mathbf{B}_2 \cdot \hat{\mathbf{n}} = \mathbf{B}_1 \cdot \hat{\mathbf{n}} \tag{4}$$

and

$$\mathbf{B}_2 - \mathbf{B}_1 = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \tag{5}$$

where $\mathbf{K}$ is the surface current at the given point. As we shall see later, in some cases we may use the above boundary conditions to claculate the surface charge or current densitites.

C. Integral Maxwell Equations

If you have a symmetric system, then you may find it simpler to solve the *integral* version of Maxwell's equations, obtained by taking the integral of both sides of the differential equations and using mathematical properties. The simplification comes from the fact that the electric and magnetic fields in symmetric systems remain constant on specific surfaces or lines that depend on the symmetry. If this is the case, the integral version of Maxwell's equations become algebraic equations, since the fields can be factored out of integrals calculated over such symmetry surfaces or lines. Read page 95 in Griffiths Vol 1. Maxwell's integral equations can be applied for any surfaces, volumes, and closed loops that we wish. However, they are *useful* only if we choose them in a smart way.

1. Gauss's Integral Law

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{in}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho dV + \cdots \tag{6}$$

In the above equation, the integral of the electric field on the left hand side is calculated over any closed surface S that we choose (surface integral). In practice, we choose the surface S dictated by the symmetry of the physical system, so that we can factor out of the integral the electric field component along the direction of $d\mathbf{S}$ out of the integral (see worked examples). For example, in the case of spherical or cylindrical symmetry, we can use the above equation to calculate $E_r(r)$ if we choose S to be a sphere or cylinder respectively. Note that the right hand side must count *all* charges enclosed by the above surface, due to ρ, σ, λ, point charges, everything. Charge is calculated in terms of charge densities as discussed in the previous section. The volume integral of ρ is calculated over the entire volume V enclosed by the surface S. The vector $d\mathbf{S}$ has direction perpendicular to the surface and points outward. Its magnitude is equal to the elementary area dS (we split the surface into small parts of area dS and sum over them). More mathematical details about such surface integrals may be found in pages 42-46 of Griffith's book.

2. *Faraday's Integral Law*

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

The left hand side integral is calculated over a closed loop, which we split into infinitesimal segments of length dl and then sum the dot product $\mathbf{E} \cdot d\mathbf{l}$ over all of them. The vector $d\mathbf{l}$ is tangential to the closed line of interest. We can transverse this closed line by following any one of the two possible directions; $d\mathbf{l}$ points along the direction of choice. The two opposite directions give opposite signs of the above line integral. However, once we have chosen the direction of $d\mathbf{l}$, the direction of $d\mathbf{S}$ that appears in the integral on the right hand side is fixed by the right hand rule, discussed by Griffiths on page 48. The right hand side integral is calculated over *any* open surface, with any possible shape, bounded by the closed line that we chose when calculating the integral on the left hand side. Unlike in Gauss' law, here the surface integral is calculated over an open surface. All the mathematical details are discussed in pages 47-53 of Griffiths. Note that Gauss's law for the magnetic field discussed next implies that any surface that we choose to calculate the surface integral on the right hand side will produce the same result, provided only that it is bounded by the closed loop used on the left hand side.

In the case of a time–independent system, we obtain from the above equation that

$$\int_C \mathbf{E} \cdot d\mathbf{l} = 0$$

for any closed loop. This is easy to solve in the case of spherical symmetry if we take the loop C to be the ϕ=constant, r=constant (C_θ) or the θ=constant, r=constant (C_ϕ) circle. In this case,

$$\int_{C_\phi} \mathbf{E}(r) \cdot d\mathbf{l} = E_\phi(r) L_{C_\phi} = 0$$

where L_{C_ϕ} is the length of the circle, which immediately gives that

$$E_\phi = 0,$$

while similarly

$$\int_{C_\theta} \mathbf{E}(r) \cdot d\mathbf{l} = E_\theta(r) L_{C_\theta} = 0$$

which immediately gives that

$$E_\theta = 0.$$

We thus conclude that, in a time–independent spherically symmetric system,

$$\mathbf{E}(\mathbf{r}) = E_r(r)\hat{r},$$

where $E_r(r)$ can be obtained either from Eq.(6) or by solving the differential equation plus the boundary conditions discussed below.

In the case of cylindrical coordinates, we obtain using the same loop as above that $E_\phi = 0$. We also obtain that $E_z = 0$ from $\nabla \times \mathbf{E}$=0, which after substituting the expressions found in Griffiths and noting that $\mathbf{E}$ only depends on r gives E_z=constant (the same result can be obtained from the integral equation), while for $r \to \infty$ E_z=0.

3. Gauss's Integral Law for the Magnetic field

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

This applies for any closed surface that we choose. In the case of spherical or cylindrical symmetry, by applying the above equation for S taken as a sphere or cylinder respectively, the above equation gives B_r=0.

4. Ampere–Maxwell Integral Law

$$\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \epsilon_0 \mu_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{S}$$

where

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} + \cdots$$

includes all currents that flow through the loop C, due to $\mathbf{J}$, $\mathbf{K}$, I, etc. Similar to Faraday's law, the line integral on the left hand side can be calculated over any closed loop that we choose, while the surface integral on the right hand side is calculated over any open surface bounded by this closed loop, with the same conventions about positive and negative directions.

D. Superposition

What happens if the system is not symmetric? Ask: can I break down the full problem into a superposition of symmetric problems? What this means is, can I express the nonsymmetric charge and current distributions as a sum of symmetric distributions, solve each symmetric problem separately, and then add the fields obtained this way to get the full solution? This is the case since Maxwell's equations are *linear*, i.e. only involve the first power of the unknown fields. The simplest symmetric system involves a point charge, and you can always separate a continuous charge or current distribution into infinitesimal charges within infinitesimal volumes (see Griffiths Vol 1 pages 83-87), but you may be able to do better.