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I. SIMPLIFYING THE MATH: ELECTRIC AND MAGNETIC POTENTIALS

Maxwell's equations are four coupled differential equations, which must be solved for six unknown functions $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. So, mathematical tricks are required to reduce the complexity. We reduce the number of coupled differential equations and unknown functions by solving for the electric and magnetic potential rather than the electric and magnetic field.

A. Definition of Electric Scalar and Magnetic Vector Potential

We first note that not every function $\mathbf{B}(\mathbf{r},t)$ can describe a magnetic field: it must satisfy the condition $\nabla\cdot\mathbf{B}$=0. From vector calculus we know that the above condition is satisfied automatically if we seek a solution $\mathbf{B}$ of the form

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t), \tag{1}$$

substitute in Maxwell's equations, and solve for $\mathbf{A}(\mathbf{r},t)$ rather than for $\mathbf{B}(\mathbf{r},t)$. In this way we have one less equation to worry about: Gauss's law for the magnetic field is automatically satisfied and we can forget about it. Eq.(1) defines the *magnetic vector potential* $\mathbf{A}(\mathbf{r},t)$.

Substituting Eq.(1) in Faraday's equation we obtain

$$\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}(\nabla\times\mathbf{A})\rightarrow\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0. \tag{2}$$

From vector calculus we know that the above condition is satisfied automatically if we seek a solution $\mathbf{E}$ of the form

$$\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla V(\mathbf{r},t)\rightarrow\mathbf{E}(\mathbf{r},t)=-\nabla V(\mathbf{r},t)-\frac{\partial\mathbf{A}}{\partial t}. \tag{3}$$

where $V(\mathbf{r},t)$ is a *scalar* function. Eq. (3) defines the *electric scalar potential* $V(\mathbf{r},t)$. The above defined $\mathbf{A}$ and V determine uniquely $\mathbf{E}$ and $\mathbf{B}$ in any general system, time–dependent or not time–independent. Rather than solving for the six functions $\mathbf{E}$ and $\mathbf{B}$, we can solve

for the four functions $\mathbf{A}$ and V and then use Eqs. (1) and (3) to obtain the fields. The fields have physical meaning, since they determine the observable force via the Lorentz force equation, the potentials are tools introduced for mathematical convenience.

The differential equations for the potentials are obtained from the remaining two Maxwell's equantions. From Gauss's law for the electric field we obtain that

$$\nabla^2 V(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\epsilon_0} - \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) \tag{4}$$

and from the Ampere–Maxwell equation we obtain

$$\nabla\times(\nabla\times\mathbf{A}(\mathbf{r},t)) = \mu_0\mathbf{J}(\mathbf{r},t) - \mu_0\epsilon_0\nabla\left(\frac{\partial V(\mathbf{r},t)}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{\partial t^2} \tag{5}$$

which after using the vector calculus property

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

and the "coincidence" in the values of the constants $\epsilon_0\mu_0 = 1/c^2$, where c is the speed of light, becomes

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{\partial t^2} = -\mu_0\mathbf{J} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right). \tag{6}$$

The problem of solving the *four* coupled Maxwell's equations for *six* unkonwn functions $\mathbf{E}$ and $\mathbf{B}$ is completeley equivalent to solving *two* coupled differential equations, Eqs.(4) and (6), for *four* unkown functions V and $\mathbf{A}$. You may recognize in Eq.(6) the form of the wave equation, whose solution gives propagating waves with speed c. But this later. Let us first deal with time–independent problems.

B. Potentials in time–independent systems

The above equations for the potentials simplify significantly if the charge and current distributions do not change with time, in which case the solutions will be time–independent. Setting all time derivative to zero, the two above equations decouple and simplify.

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{7}$$

is called Poisson's equation. It reduces the problem of calculating the electric field, a vector function with three unknown components determined by two equations that couple these three components in the general system, to solving one differential equation for a single *scalar*

unknown function V. We no longer have to worry about three electric field components, just one scalar function.

To obtain the magnetic field, we unfortunately must still solve for a vector function $\mathbf{A}$, so at first glance there is no clear advantage of $\mathbf{A}$ over $\mathbf{B}$. The mathematical advantages of $\mathbf{A}$ become more clear when dealing with time–dependent problems and electromagnetic waves. For static problems, $\mathbf{A}$ satisfies an equation of the Poisson form, formally similar to the one satisfied by V, which allows us to draw on the intuition developed for electrostatic problems. We note that we can find an infinite number of functions $\mathbf{A}(\mathbf{r}, t)$ all of which give the same $\mathbf{B}$ via Eq.(1). In particular, if a vector function $\mathbf{A}$ satisfies Eq.(1), then the vector function $\mathbf{A}' = \mathbf{A} + \nabla\lambda(\mathbf{r}, t)$, where λ is *any* scalar function, also gives the same magnetic field, $\nabla \times \mathbf{A} = \nabla \times \mathbf{A}' = \mathbf{B}$. Recalling that $\mathbf{A}$ does not have any physical meaning other than describing the correct magnetic field via Eq.(1), it is clear that we have a choice in $\mathbf{A}$. We will make this choice to simplify the mathematics. A vector function $\mathbf{A}$ is uniquely determined if we know the values of both $\nabla \times \mathbf{A}$ (here fixed by $\mathbf{B}$) *and* $\nabla \cdot \mathbf{A}$. So we choose the value of $\nabla \cdot \mathbf{A}$ that simplifies the differential equation Eq.(6). For a time–independent system, it is clear that the simplest $\mathbf{A}$ is determined by the differential equations

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}(\mathbf{r}). \tag{8}$$

and

$$\nabla \cdot \mathbf{A} = 0, \tag{9}$$

where the latter condition is called the *Coulomb gauge*.

In an arbitrary coordinate system, the Laplacian of a vector function, which enters in Eq.(8), is defined by

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}), \tag{10}$$

where the expressions for $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ in different coordinate systems can be found in the cover of Griffiths. In the particular case of cartesian coordinates,

$$\nabla^2 \mathbf{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z, \tag{11}$$

i.e. the intuitive result that the Laplacian of $\mathbf{A}$ has coordinates equal to the Laplacian of the coordinates of $\mathbf{A}$ holds. It is OK to obtain $\nabla^2 A_i(\mathbf{r}, t)$, i=x,y,z, by expressing $\mathbf{r}$ in a different coordinate system using the expressions in the cover of Griffiths. However, we

must do this for $A_i(r,\phi,z)$ (cylndrical) or $A_i(r,\theta,\phi)$ (spherical) with i=x,y,z, not i=r,ϕ,z or i=r,θ,ϕ. A simple expression analogous to Eq.(11) does not hold for the components of $\mathbf{A}$ in an arbitrary coordinate system, although Eq.(10) is always correct.

So what *is* the advantage of solving for the vector function $\mathbf{A}$ instead of the vector function $\mathbf{B}$? Unlike for the Ampere–Maxwell equation for $\mathbf{B}$, which couples the different B–field components in $\nabla \times \mathbf{B}$, Eq.(8) reduces to three *independent* Poisson differential equations for the x, y, and z components of $\mathbf{A}$. Thus, the three unknown cartesian components decouple in Eq.(8), unlike for B_i in the Ampere–Maxwell equation. Note however that we must still satisfy the condition Eq.(9). Another advantage of Eq.(8) is the mathematical resemblance between the electric and magnetic potential problems, as both potentials are obtained by solving differential equations of the Poisson form:

$$\nabla^2 A_i(\mathbf{r}) = -\mu_0 J_i(\mathbf{r}) \ , \ i = x,y,z \ \leftrightarrow \nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{12}$$

Since the same differential equation has the same solution, independent of the different physical meaning of the quantities involved, we can make an analogy to the electrostatic problem, for which we have better physical intuition, and note that A_i will have the same mathematical form as V if we formally substiture

$$\epsilon_0 \rightarrow 1/\mu_0 \ , \ \rho \rightarrow J_i \ , \ i = x,y,z. \tag{13}$$

We also see that, in a time–independent system, the electric and magnetic fields decouple: charges create electric fields and currents create magnetic fields. No currents no magnetic fields, no *net* charge no electric field (recall however our earlier discussion that we can get current in a neutral system).

We have reduced the problem of calculating the electromagnetic field in any time–independent system with given charge and current distribution to the solution of the scalar Poisson differential equation. For this we also need boundary conditions. From the continuity of the parallel electric field components across any sufrace (Griffiths page 116), we obtain that the potential V is a *continuous function everywhere in space*, independent of whether we have boundary surfaces, charged or not:

$$V_1 = V_2 \tag{14}$$

at every point on a surface, where 1 and 2 denote the two regions of space separated by the surface. From the discontinuity of the electric field component perpendicular to a surface

with surface charge density σ (Griffiths page 116–117), we obtain for any point along the surface

$$\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} = -\frac{\sigma}{\epsilon_0}, \tag{15}$$

where $\mathbf{n}$ is the unit vector perpendicular to the surface at the point of interest, pointing from region 1 to region 2, and we defined the *perpendicular derivative*

$$\frac{\partial V}{\partial n} = \nabla V \cdot \mathbf{n}. \tag{16}$$

The two boundary conditions Eq. (15) and Eq. (14) apply at every point across any surface of any shape or charge and are derived in Griffiths pages 114–117 from Gauss's and Faraday's laws for the electric field. From the theory of differential equations, the solution of Poisson's equation within a volume enclosed by a closed surface S is *unique* provided that we know (i) the value of V at all points on the surface, or (ii) the value of the perpendicular derivative $\partial V/\partial n$, Eq. (16), at all points on the surface, or (iii) the value of V at some points and of $\partial V/\partial n$ at the rest of the points of the surface. We also consider as additional boundary conditions the behavior of the potential at special points such as infinity or $\mathbf{r}$=0, where the solution of the differential equation has to reproduce the behavior expected when viewing the charges while sitting at these special points. For example, when viewing a charge distribution from infinity, its finite dimensions do not matter and we can approximate them as zero, thus viewing the charge distribution from a large distance as a point or surface charge.

If you somehow, anyhow, find a function $V(\mathbf{r})$ that satisfies both Poisson's equation and the above boundary conditions, then this function is necessarily the one and only solution. Just find it, using your physical intuition, your gut feeling, systematic procedures, anything! If we somehow know the value of V everywhere on the boundary surface, we do not need the boundary condition Eq.(15) to solve the differential equation. Eq.(15) is still useful however as we can use it to obtain the unknown σ.

In many cases, we have incomplete knowledge of the charge distributions that create the electromagnetic fields, which are supposed to be a known input to Maxwell's equations. For example, we may know $\rho(\mathbf{r})$ in the interior of a closed surface S, while ρ is unknown outside the surface S. In this case, we replace our incomplete knowledge of the charge distribution outside S by knowledge, in some way, of the potential or its perpendicular derivative values everywhere on the surface S. We do not know the charges outside S but we know that they are such that they give, in combination with the known $\rho(\mathbf{r})$, fixed known values of

the potential or the electric field everywhere on S. In such a case, the above information is enough to obtain $V(\mathbf{r})$ (and from this the electric field) everywhere *inside* S, while V remains unknown outside S (but we do not care what happens there). For this, we take advantage of the mathematical theorem that states that the solution of the Poisson differential equation inside a volume enclosed by a surface S is *unique* if we know either V or $\mathbf{E} \cdot \mathbf{n}$ everywhere on the surface S, via some other means. In practice, the above concept is very useful for example when dealing with conductors. As discussed by Griffiths pages 125–128, every point on the surface or the interior of a perfect conductor in electrostatic equilibrium has the same potential value. The value of this potential may be deduced, for example, if the conductor is connected to a battery (it is the battery voltage).

In the case of the vector potential $\mathbf{A}$, obtained from the three Poisson equations Eq.(8), the boundary conditions on any surface S are derived in Griffiths page 307. All three components of $\mathbf{A}$ must be *continuous* across any surface S separating regions 1 and 2, i.e. everywhere in space:

$$\mathbf{A}_1 = \mathbf{A}_2. \tag{17}$$

On the other hand, the perpendicular component of $\mathbf{A}$ is discontinuous. If the unit vector $\mathbf{n}$ perpendicular to the surface points from region 1 to region 2, then

$$\nabla \times \mathbf{A}_2 - \nabla \times \mathbf{A}_1 = \mu_0(\mathbf{K} \times \mathbf{n}), \tag{18}$$

a boundary condition that is often useful for calculating the surface current $\mathbf{K}$.

Let us now establish a relationship, in the case of a time–independent system, between the V defined mathematically by Eq.(3) and the potential that you learned in previous courses. From the definition Eq.(3), we obtain by integration that, for any two points A and B and for any path between them

$$V(\mathbf{r}_B) - V(\mathbf{r}_A) = -\int_A^B \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l}.$$

Noting from Faraday's integral equation that, for a time–independent system, $\int \mathbf{E} \cdot d\mathbf{l}$=0 over any closed loop, we see that the line integral $\int_A^B \mathbf{E} \cdot d\mathbf{l}$ has the same value for any path that we choose to follow to go from A to B. This result implies that, for a time–independent system, the following function of position $\mathbf{r}$ satisfies the definition of the potential, Eq.(3):

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\mathbf{r}_O} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}',$$

where point $\mathbf{r}_O$ is an arbitrary point that we choose to assign $V(\mathbf{r}_O) = 0$. Of course, the above equation gives V for a given $\mathbf{E}$, while we introduced V to calculate the unknown $\mathbf{E}$.

We note here that the functions $V(\mathbf{r})$ or $V(\mathbf{r}) + c$, where c is any constant, give the *same exact* electric field. The electric potential function that describes a physical system is not uniquely defined. Only differences in potentials between two points enter in the calculation of any physical quantity, so we can choose the point O where $\Phi(\mathbf{r}_O) = 0$ arbitrarily, whatever simplifies the math.

C. Potential Energy

The concept of energy conservation is very useful for analyzing complex situations. A time–independent electric field exerts a conservative force on a charged particle, and thus we can associate an electrostatic potential energy with the position of a charged particle within this field. If I want to bring in a charge from infinity, where the electrostatic forces are zero, I must exert a force that compensates the electrostatic force. In the process I do work, which is transformed into potential energy stored in the system. This potential energy is equal to the work that I do when I bring in the charge at constant velocity, in which case I exert a force that compensates the electrostatic force $q\mathbf{E}(\mathbf{r})$. The kinetic energy then remains the same, and all the work done by the above force goes into potential energy:

$$U(\mathbf{r}) - U(\mathbf{r}_0) = -q \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r}$$

is well–defined in the case of static systems, where $\partial \mathbf{B}/\partial t = 0$, as can be seen from Faraday's equation. We note that, in a time–independent system, the potential energy is proportional to the electric potential Eq.(3), which was introduced in order to simplify the solution of the differential equations:

$$U(\mathbf{r}) = qV(\mathbf{r}).$$

This coincidence gives physical meaning to the mathematical quantity V. A similar intuitive interpretation is not available for the vector potential $\mathbf{A}$ that gives the magnetic field.

Let us now calculate the total potential energy stored in a region of space where a charge density $\rho(\mathbf{r})$ is located. This is equal to the work required to overcome the Coulomb repulsion and create a static charge distribution. We always talk about the potential energy of the entire electric field+charge system. We consider a time–independent system. The total

potential energy stored in a system with N point charges q_i, i=1, $\cdots$ N is given by

$$E = \sum_{i=1}^{N} q_i \sum_{j=1}^{i-1} V_j(\mathbf{r}_i) = \frac{1}{2} \sum_{i \neq j} q_i V_j(\mathbf{r}_i)$$

where V_j is the potential due to charge q_j. The factor 1/2 was added to avoid double counting: the potential energy $q_i V_j$ applies to a pair of charges q_i and q_j and is counted twice in the above sum as $q_i V_j = q_j V_i$. From the above equation we see that, in the continuum limit, the energy E equal to the work done in order to assemble a charge distribution $\rho(\mathbf{r})$, is given by

$$E = \frac{1}{2} \int V(\mathbf{r}) \; dq \tag{19}$$

where the integral is calculated over all charges, including surface or line distributions: $dq = \rho dV, \sigma dS, \lambda dl$. In the case of a volume charge density ρ,

$$E = \frac{1}{2} \int V(\mathbf{r}) \rho(\mathbf{r}) d\mathbf{r} \tag{20}$$

which after using Gauss's law to express ρ in terms of the electric field becomes

$$E = \frac{\epsilon_0}{2} \int [\nabla \cdot \mathbf{E}(\mathbf{r})] V(\mathbf{r}) d\mathbf{r}.$$

We now simplify the above expression by using the derivative property

$$\nabla \cdot (\mathbf{E}V) = (\nabla \cdot \mathbf{E})V + \mathbf{E} \nabla V,$$

the definition of the potential $\mathbf{E} = -\nabla V$, and Gauss's theorem that relates volume and surface integrals:

$$E = \frac{\epsilon_0}{2} \left[\int E^2(\mathbf{r}) d\mathbf{r} + \int_S V(\mathbf{r}) \; \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} \right],$$

where the volume integral can be performed over the volume enclosed by any closed surface S, provided that this volume includes the entire charge. If we increase the volume to include the entire space, then the surface integral vanishes since, at $r \to \infty$, $E \propto 1/r^2, V \propto 1/r$, and the surface $\propto r^2$. We thus finally obtain that

$$E = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2(\mathbf{r}) d\mathbf{r}. \tag{21}$$

This expression for the total energy stored inside a time–independent system gives E in terms of the electric field caused by $\rho(\mathbf{r})$, which we assume has been calculated. We can think of the total energy as stored inside the field with an energy density of $\frac{\epsilon_0}{2} E^2(\mathbf{r})$. We note that we cannot use the principle of superposition to add up the energies of two subproblems to obtain the energy of the full problem since the energy Eq.(21) depends on $\mathbf{E}^2$. Eqs. (20) and (21) provide two alternative ways for calculating the energy stored in a static system.

D. Energy conservation in a time–dependent system

As discussed in Griffith's pages 71–73, the above expressions for the energy stored inside the electromagnetic field can be generalized to a time–dependent general system. Consider a volume V in space enclosed by a surface S. Assume that you have calculated the electric and magnetic fields **E** and **B** everywhere inside this volume, due to charges and currents both inside and outside the volume. Using the Lorentz force, the work performed, during a time interval dt, on a charge $dq = \rho dV$ moving with velocity **v** inside an electromagnetic field is given by

$$dW = \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \mathbf{v} dt = dq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt = \mathbf{E} \cdot \mathbf{v} \rho dV dt = \mathbf{E} \cdot \mathbf{J} dV dt \tag{22}$$

where we used the relation $\mathbf{J}=\rho\mathbf{v}$. The total work per unit time (i.e. the power) done on all charges inside volume V is then given by

$$\frac{dW}{dt} = \int_V \mathbf{E}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) dV \tag{23}$$

As shown in Griffiths page 72, by using Maxwell's equations to transform $\mathbf{E} \cdot \mathbf{J}$ one can derive the Poynting theorem that expresses energy conservation in a general time–dependent system:

$$\frac{dW}{dt} = -\frac{d}{dt} \int_V U dV - \int_S \mathbf{N} \cdot d\mathbf{S} \tag{24}$$

where

$$U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \tag{25}$$

is the energy per unit volume (energy density) stored in the electromagnetic field inside the volume V, and

$$\mathbf{N} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \tag{26}$$

is Poynting's vector and describes the energy per unit time per unit surface that is transported by the fields: $\mathbf{N} \cdot d\mathbf{S}$ is the energy per unit time that passes through dS. The above energy conservation equation can also be expressed in differential form at every point in space:

$$\nabla \cdot \mathbf{N} + \frac{\partial U}{\partial t} = -\mathbf{E} \cdot \mathbf{J}. \tag{27}$$