E-M I, Department of Physics, University of Crete

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I. SOLVING THE POISSON EQUATION

We solve Poisson's equation, of the general form $\nabla^2\Phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0$, everywhere inside a *closed* volume V. This volume must be completely enclosed by a surface S. Parts of the closed surface S may extend to infinity. In general, we only know $\rho(\mathbf{r})$ inside S, i.e. at every point $\mathbf{r}$ of the volume V where we seek a solution $\Phi(\mathbf{r})$. ρ may be *unknown* outside S, where we do not care what $\Phi(\mathbf{r})$ is. Therefore, we cannot use the superposition principle since we do not know the charges and currents everywhere in space. We could use the differential form of Maxwell's equations, but then we need to know the surface charges (or surface currents) everywhere on S, as well as the electric and magnetic fields right outside S, in order to obtain the necessary boundary conditions for the fields. However, we can substitute our ignorance of the full input to Maxwell's equations by the knowledge, through some other means, of the values of Φ or its perpendicular derivative everywhere on the surface S.

We first divide V into regions 1, 2, $\cdots$, N. We then write down the general solutions $\Phi_1(\mathbf{r})$, $\Phi_2(\mathbf{r})$, $\cdots$, $\Phi_N(\mathbf{r})$, of the differential equation separately in each region, if possible. Unknown constants enter into the above solutions of the differential equation, whose values *differ in each of the above separate regions.* The values of these constants are determined by the boundary conditions (i) at each dividing surface with known surface charge and current (as discussed in the previous lecture), (ii) at the boundary surface S that encloses the entire volume V of interest. If S extends to infinity, we must make sure that the solution has the correct behavior at infinity, i.e. it reproduces the result expected for charge and current distributions with negligible dimensions, since these finite dimensions do not matter when we look at the charges and currents from an infinite distance. We must also make sure that the solution reproduces the result expected for a point or line charge or a line current in the immediate vicinity of such charges or currents. Φ will become infinite there, the question is how. Finally, we must make sure that the solution is finite at $\mathbf{r}$=0 if there is no point or line charge or current there. The above determine the full boundary conditions dicated by the physics of the problem. We must always keep in mind that the differential solution has an infinite number of solutions, but only one of these solutions describes the given physical system: the fields are described by uniquely defined functions for a given system. The boundary conditions determine which solution of the differential equation is physical.

Below we list some general ideas to guide you in picking the right approach to solving the Poisson equation for a particular system. One thing to *always* remember: once you guess or find, *in whatever way*, one solution of the differential equation that also satisfies all boundary conditions, then this solution is the correct one, the only solution. This can be shown from the mathematical properties of the Poisson differential equation (uniqueness of the solution, see e.g. Griffiths volume 1 pages 144–156). Note that it may not be possible to find the general solution of the Poisson equation in closed form for your given system. If all of the ideas outlined below fail, your only way out may be to solve the Poisson equation numerically using the computer. You can learn some numerical methods for doing that on a computer in the Computational Physics course. Before getting there however, try to exhaust your physical intuition, imagination, and knowledge of some general concepts and tricks, in the spirit of Sherlock Holmes or a medical doctor that examines his/her patient before deciding what tests to do.

To determine the boudary conditions, it is useful to remember the following general properties of a **conductor** in *steady state* (i.e. following an initial time–dependent regime as short as several femto–seconds, during which the charges and currents reach equilibrium):

1. $\mathbf{E}$=0 in the interior of a conductor, either because $\mathbf{E} = \mathbf{J}/\sigma$ (Ohm's law, σ=conductivity) and $\sigma \to \infty$ for negligible resistance (like when shorting a circuit), or because there are no batteries involved to provide energy and in the *steady state* we must have $\mathbf{E}$=0 so that $\mathbf{J} \cdot \mathbf{E}$=0 and there is no work done on the moving charges that would require an energy source.

2. $\rho(\mathbf{r})$=0 everywhere inside a conductor ($\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\mathbf{E} = 0$). This does not mean that there is no charge inside a conductor, it just means that the positive charge cancels out the negative charge everywhere inside.

3. If the conductor is charged, then its entire charge accumulates on its surface, described by a surface charge density σ.

4. $V(\mathbf{r})$=V=constant everywhere inside the conductor and on its surface ($\mathbf{E} = 0 = -\nabla V$).

5. The electric field right outside the conductor is perpendicular to its surface (due to the boundary conditions that the tangential component of **E** is continuous on the surface, while **E**=0 inside the conductor).

6. We can use Gauss's integral equation $\int \mathbf{E} \cdot d\mathbf{S} = Q_{int}/\epsilon_0$ plus the fact that **E**=0 inside a conductor to determine the charge induced when bringing in an external charge.

A. Symmetric systems

Is there some symmetry in the system? If yes, then $V(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ must reflect it. For example, in a spherically symmetric system, we seek a solution of the Poisson equation of the form $V(r)$, in a cylindrically symmetric system we seek a solution of the form $V(\rho)$, and in the case of translational invariance along the x-y plane we seek a $V(z)$. These are properties dictated by the symmetry of the physical system and obeyed by the physically correct solution of the differential equation. By using the expressions of the Laplacian $\nabla^2 V$ in the cover of Griffiths and noting that, in a symmetric system, it is convenient to work in the appropriate coordinate system, we reduce Poisson's equation to the following ordinary differential equations:

$$\frac{d}{dr}\left(r^2 \frac{dV(r)}{dr}\right) = -\frac{r^2 \rho(r)}{\epsilon_0} \tag{1}$$

for a spherically–symmetric system,

$$\frac{d}{dr}\left(r \frac{dV(r)}{dr}\right) = -\frac{r \rho(r)}{\epsilon_0} \tag{2}$$

for a cylindrically–symmetric system, while for translational invariance along the x–y plane

$$\frac{d^2 V(z)}{dz^2} = -\frac{\rho(z)}{\epsilon_0}. \tag{3}$$

We then solve these ordinary differential equations in each separate region of space and determine the constants that enter by using the boundary conditions for the potential or its derivative.

The easiest thing to try when dealing with a fully symmetric system is to use Maxwell's integral equations. However, the latter require knowledge of the charges and currents that we may no have. The above *ordinary* differential equations apply to a fully symmetric system and give the solution even if we do not know all charges and currents, provided that we know the proper boundary conditions. The *partial* differential equation is the only way to proceed if the system is not fully symmetric, in which case the integral equations are not useful in practice.

B. Not quite symmetric? Superposition Principle?

What if there is no symmetry? One way to proceed is to use the general property of superposition that arises from the linearity of the differential equations. If we split up the charge distribution $\rho(\mathbf{r})$ into two (or more) distributions $\rho_n(\mathbf{r})$, chosen at will with the constraint that

$$\rho(\mathbf{r}) = \sum_{n=1}^{N} \rho_n(\mathbf{r}) \tag{4}$$

(same for any surface or line charge distributions), then we can first solve separately the N Poisson equations, one for each ρ_n while ignoring all the other ρ_i, $i \neq n$ and obtain the corresponding solutions $V_n(\mathbf{r})$, n=1,$\cdots N$. Due to the linearity of the differential equation, the solution $V(\mathbf{r})$ for the full ρ is then obtained by simply summing up the above V_n, no further work required:

$$V(\mathbf{r}) = \sum_{n=1}^{N} V_n(\mathbf{r}). \tag{5}$$

Be smart and exploit this property to reduce complicated problems, such as problems without symmetry that require solving a partial differential equation, into simpler ones (e.g. symmetric problems).

For example, a charge distribution ρ can be thought of as a superposition/sum of infininitesimal charges $\rho(\mathbf{r}')d\mathbf{r}'$ located at every point $\mathbf{r}'$ in space. Each such infinitesimal charge creates a potential $V_{\mathbf{r}'}(\mathbf{r})$ at point $\mathbf{r}$, whose distance from $\mathbf{r}'$ is $|\mathbf{r}-\mathbf{r}'|$:

$$V_{\mathbf{r}'}(\mathbf{r}) = \frac{\rho(\mathbf{r}')d\mathbf{r}'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

From the principle of superposition we obtain that $\rho(\mathbf{r})$ creates the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{7}$$

which we calculate by doing the integral, and then obtain the electric field from $\mathbf{E} = -\nabla V$. Note that the above equation only applies for a charge distribution that *vanishes at infinity*, since we assumed $V = 0$ at infinity in Eq.(6). Similar integral expressions for the potential can be obtained for surface, line, or point charge distributions:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}')dS'}{|\mathbf{r}-\mathbf{r}'|}\ ,\ V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_l \frac{\lambda(\mathbf{r}')dl'}{|\mathbf{r}-\mathbf{r}'|}\ ,\ V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_n \frac{Q_n}{|\mathbf{r}-\mathbf{r}_n|}. \tag{8}$$

Although Eq.(7) provides the solution without the need to solve a differential equation, in practice the integral is often hard to calculate, while in many cases we do not know $\rho(\mathbf{r})$ at every position in space. This is why we solve the Poisson differential equation. Note that Eq.(7) is much simpler to calculate as compared to the analogous integral expression of the electric field as superposition of Coulomb electric fields,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'. \tag{9}$$

In Eq.(9) we must add up vectors rather than scalars and we must worry about components.

A similar superposition scheme can be used to calculate the magnetic field. Since the three cartesian components of the vector potential $\mathbf{A}$ are given by three Poisson equations similar to the one satisfied by V, we obtain from the above expression of the Poisson equation solution that

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{J_i(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{10}$$

which only applies if $\mathbf{J}$ vanishes at infinity. Similar expressions hold for the vector potential created by linear or surface currents:

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{K_i(\mathbf{r}')dS}{|\mathbf{r}-\mathbf{r}'|}\ ,\ A(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|}. \tag{11}$$

The above expressions satisfy the gauge condition $\nabla\cdot\mathbf{A}=0$ so we are OK. From the above we see immediately that the direction $\mathbf{A}$ coincides with the direction of the current flow, which is very useful for guessing the direction of the magnetic field, or if we decide to calculate $\mathbf{A}$ by solving the Poisson equations. By calculating $\nabla\times\mathbf{A}$ using the above expressions for $\mathbf{A}$ we obtain Biot–Savart's law (Griffiths page 272).

C. No symmetry? Method of Images?

So if everything fails what next? Let us consider a general system. We need the solution $V(\mathbf{r})$ of the Poisson differential equation inside a volume V enclosed by a surface S. This solution depends on (i) the known charge distribution $\rho(\mathbf{r})$ *inside* S (ii) the known boundary condition for the potential on S. We can simplify the full problem by breaking it into a superposition of two sub–problems, $\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r})$, as follows.

First, we consider the effect of the charge density. We do not know the charge outside the closed surface S, only inside. We consider a ρ_1 equal to the known ρ inside the surface S and equal to any convenient distribution that we wish outside S (or zero). ρ_1 is then known everywhere and we can use Eq.(7) (or solve the Poisson equation) to calculate the potential V_1 due to the known ρ_1 over the entire space.

Second, we consider $\rho_2 = \rho - \rho_1$. Recalling that, inside S, ρ_1 coincides with the known distribution ρ, we obtain that ρ_2=0 everywhere *inside* S; outside S it is unknown since the original ρ is unknown there. However, we do not need to obtain the potential outside S, while inside $\nabla^2 V_2$=0. We can replace our ignorance of ρ_2 outside S by the proper boundary condition, i.e. the potential on S in the original problem (which is given) minus the potential on S calculated for the ρ_1 above.

The first step above simplifies since we do not have to consider a boundary condition, while we try to choose a simple ρ_1. The second step simplifies since $\rho_2 = 0$ inside S. The method of images, described in Griffith's vol 1 pages 156–164, is an example of how the some particular non–symmetric time–independent systems can be described by using the above general scheme.

D. Only boundary conditions? Solve Laplace's equation using separation of variables?

The idea is (i) express the solution inside S as a product of three functions of each of the three coordinates in the chosen coordinate system, (ii) substitute this product form in the differential equation and obtain the general form of the three product functions of one variable by solving an ordinary differential equation, and (iii) try to satisfy the boundary conditions. The method works in a coordinate system dictated by the geometry of the physical system. In this coordinate system the potential $\Phi(\mathbf{r}) = \Phi(s_1, s_2, s_3)$, where (s_1, s_2, s_3)=(x, y, x) or (r, θ, ϕ) or (ρ, ϕ, z) etc, depending on the coordinate system that is most convenient to use for the given geometry of the boundary S. Inside S, $\rho = 0$, and we try to see if we can find solutions that have the form

$$\Phi(\mathbf{r}) = \Phi(s_1, s_2, s_3) = \Phi_1(s_1)\Phi_2(s_2)\Phi_3(s_3). \tag{12}$$

We substitute this form into the Laplace equation $\nabla^2\Phi=0$, using the front page of Griffiths to express ∇^2 in the coordinate system (s_1, s_2, s_3), and try to obtain the general form of the different Φ_1,Φ_2, and Φ_3 that satisfy the differential equation inside S. We then express the solution Φ in the form

$$\Phi(\mathbf{r}) = \sum_j c_j \Phi_1^j(s_1)\Phi_2^j(s_2)\Phi_3^j(s_3) \tag{13}$$

where the (finite or infinite) sum is over all the possible solutions of the differential equation. Finally, the unknown coefficients c_j are obtained by making sure that the above expression satisfies the boundary conditions. Below we give some specific examples.

1. *Cartesian coordinates, two dimensions*

In this case, S consists of planes perpendicular to the cartesian axes and there is symmetry along z so that

$$\Phi = \Phi(x, y) = X(x)Y(y) \tag{14}$$

where X and Y are unknown functions. Substituting into the Laplace equation we obtain inside S that

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0, \tag{15}$$

a relation that must hold for every x and y. Therefore,

$$\frac{1}{X}\frac{d^2X}{dx^2} = \pm k^2 \ , \ \frac{1}{Y}\frac{d^2Y}{dy^2} = \mp k^2 \tag{16}$$

where k can be zero. We thus have the following possible trial functions:

$$\Phi(x, y) = (A_1x + A_2)(B_1x + B_2) \tag{17}$$

$$\Phi(x, y) = (A_1 \cos kx + A_2 \sin kx)(B_1 \cosh ky + B_2 \sinh ky) \tag{18}$$

$$\Phi(x, y) = (A_1 \cosh kx + A_2 \sinh kx)(B_1 \cos ky + B_2 \sin ky). \tag{19}$$

We take a sum of all or some of the above possible solutions and determine the constants by satisfying the boundary conditions. The solution is unique, so whatever works is good. We only need to consider a finite sum if (i) the boundary condition V is continuous everywhere on S, (ii) only some of the above functions enter in the boundary conditions. Otherwise, we must use Fourier's theorem (Griffith's page 169) to determine the unknown constants in an infinite sum.

2. Spherical coordinates, azimuthal symmetry

When S is spherical and, for simplicity, there is no dependence on ϕ due to symmetry, we try

$$\Phi(r,\theta) = R(r)\Theta(\theta) \tag{20}$$

and obtain that

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) = l(l+1), \tag{21}$$

where we express the unknown constant in the form $l(l+1)$ for convenience without loss of generality, and

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) = -l(l+1). \tag{22}$$

The above ordinary equations can be solved to obtain that

$$R(r) = Ar^l + \frac{B}{r^{l+1}}\ ,\ \Theta(\theta) = P_l(\cos\theta), \tag{23}$$

where P_l are the Legendre polynomials (Griffiths page 180). We express the general solution as a sum of the above produict functions with coefficients obtained by trying to satisfy the boundary condition on S.

3. Cylindrical coordinates, symmetry in z

We try

$$\Phi(r,\phi) = R(r)F(\phi) \tag{24}$$

and obtain that

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr}) = \pm m^2, \tag{25}$$

where m is any number, and

$$\frac{1}{F}\frac{d^2F}{d\phi^2} = \mp m^2 \tag{26}$$

So we have the possible solutions

$$\Phi(r,\phi) = (A_1\ln r + A_2)(B_1\phi + B_2) \tag{27}$$

$$\Phi(r,\phi) = (A_1 r^m + A_2 r^{-m})(B_1\cos m\phi + B_2\sin m\phi). \tag{28}$$